

## SPECTRAL GAPS FOR NORMALLY HYPERBOLIC TRAPPING

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ABSTRACT. We establish a resonance free strip for codimension 2 symplectic normally hyperbolic trapped sets with smooth incoming/outgoing tails. An important application is wave decay on Kerr and Kerr–de Sitter black holes. We recover the optimal size of the strip and give an  $o(h^{-2})$  resolvent bound there. We next show existence of deeper resonance free strips under the  $r$ -normal hyperbolicity assumption and a pinching condition. We also give a lower bound on one-sided cutoff resolvent on the real line.

This paper is a collection of results regarding resonance free strips (also known as *spectral gaps*) and resolvent estimates in the presence of *normally hyperbolic trapping*. Such trapping has received a lot of attention recently because of its connection with exponential decay of waves on black hole backgrounds and with exponential decay of correlations for contact Anosov flows.

In [WuZw], Wunsch and Zworski showed existence of a small spectral gap for symplectic normally hyperbolic trapped sets under the assumption that the incoming/outgoing tails  $\Gamma_{\pm}$  are smooth and have codimension 1 in the phase space; this includes subextremal Kerr and Kerr–de Sitter black holes. More recently, Nonnenmacher and Zworski [NoZw] extended this result by (a) assuming much weaker regularity of  $\Gamma_{\pm}$ , namely that their tangent spaces at the trapped set have merely continuous dependence on the base point (b) making no assumption on the codimension of  $\Gamma_{\pm}$  (c) establishing the optimal size of the gap. The assumptions of [NoZw] apply to more general situations, including contact Anosov flows.

The first result of this paper (Theorem 1) is a spectral gap of optimal size under the assumptions of [WuZw] (and the orientability of  $\Gamma_{\pm}$ ). The novelty compared to [NoZw] is the  $o(h^{-2})$  resolvent bound in this gap. The short proof presented here is also more direct than the ones in [WuZw, NoZw] because it relies on regular semiclassical analysis rather than exotic symbol calculus, at a cost of not recovering the extensions (a), (b) discussed above.

We next show (Theorem 2) the existence of deeper resonance free strips, under the additional assumptions of  $r$ -normal hyperbolicity and *pinching*. The second gap and a Weyl law for resonances in between the two gaps were previously proved in [Dy1] under the same assumptions. The present paper essentially removes the projector  $\Pi$  from the method of [Dy1], working directly with the pseudodifferential operators  $\Theta_j^+$

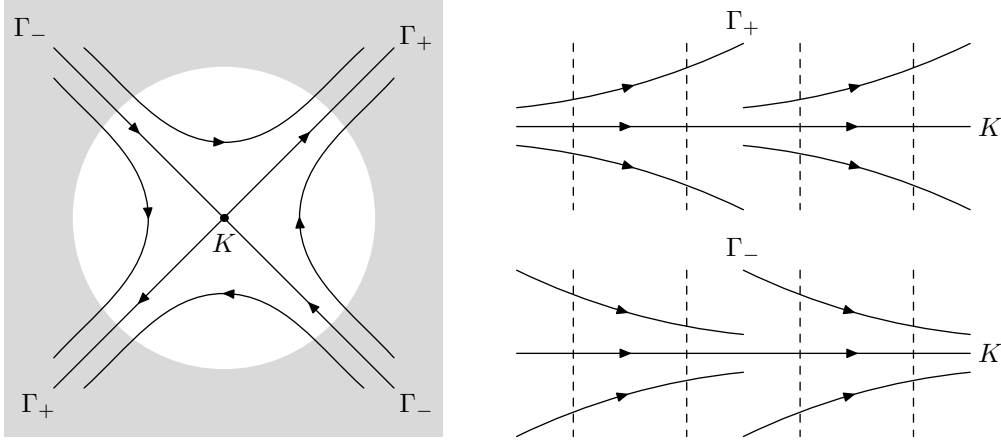


FIGURE 1. An illustration of normally hyperbolic trapping, with the Hamiltonian flow lines of  $p$  on the whole  $T^*X$  shown on the left and the restriction of the flow to  $\Gamma_{\pm}$ , on the right. The shaded region is the set  $\{q > 0\}$  where the complex absorbing operator takes over. The dashed lines are the flow lines of  $H_{\varphi_{\pm}}$ , see §2.2.

instead (see the discussion of the proof below). This shows the existence of additional resonance free strips, but does not recover the Weyl law or the structure of  $\Pi$  (which is important in understanding wave decay, see [Dy2]). For a different yet related setting of Pollicott–Ruelle resonances for contact Anosov flows, existence of multiple gaps under a pinching condition and a Weyl law for the first band of resonances was proved in [FaTs]. Our methods also bear some similarities to the recent work [DFG] on resonances for the geodesic flow on hyperbolic quotients; the horocyclic operators  $\mathcal{U}_-$  of [DFG] play the same role as the operators  $\Theta_j^+$  in this paper.

We finally show a lower bound of  $h^{-1}\sqrt{\log(1/h)}$  on the one-sided cutoff resolvent on the real line, complementing the upper bounds of [BuZw, DaVa3] – see (1.7) and §5.

**Setup.** Similarly to [WuZw], we let  $X$  be a compact manifold with a fixed volume form and study the operator

$$R(\lambda; h) := (P(h) - iQ(h) - \lambda)^{-1} : L^2(X) \rightarrow L^2(X), \quad (1.1)$$

where  $P(h), Q(h) \in \Psi_h^0(X)$  (henceforth denoted simply  $P, Q$ ) are self-adjoint semiclassical pseudodifferential operators of order 0 with principal symbols  $p, q$  (see §2 for the semiclassical notation used) and  $\lambda \in \mathbb{C}$  satisfies  $\lambda = \mathcal{O}(h)$ . We furthermore assume that  $q \geq 0$  everywhere and  $q(x, \xi) > 0$  for  $|\xi|$  large enough. The family  $R(\lambda; h)$  is meromorphic in  $\lambda$ , and we call its poles *resonances*.

One difference from [WuZw] is the fact that  $P, Q$  have order 0; this can be arranged by modifying these operators in the elliptic region and is only done here to simplify

notation. Another difference is that we require that  $R(\lambda)^{-1}$  depends linearly on  $\lambda$ ; one can reduce the general situation to this case by [Dy1, Lemma 4.3].

A typical application is when  $P = h\sqrt{\Delta}$  on some noncompact Riemannian manifold and  $Q$  is a complex absorbing potential which is elliptic as  $(x, \xi) \rightarrow \infty$ ; one can then modify  $P$  and  $Q$  in  $\{q > 0\}$  to convert to operators of order 0 on a compact manifold. Estimates on the model resolvent (1.1) imply bounds on the scattering resolvent of the original problem under additional assumptions at infinity, as shown in [DaVa1]. Our proofs also work in the general framework of [Dy1, §4.1], which applies to a variety of scattering problems; [Dy1, §§4.2, 8.1] reduces the general case microlocally to a neighborhood of the trapped set.

As a particular application, [Va] shows exponential decay of linear waves on Kerr–de Sitter black holes (modulo a finite dimensional space) using a resolvent estimate of type (1.2). We refer the reader to [Va, Dy2] for a discussion of the relation between resolvent bounds and wave decay, and for an overview of previous results on wave decay for black holes. We also remark that the analysis of trapping is critical for the understanding of semilinear and quasilinear wave equations on black holes, since a nontrapping estimate of  $\mathcal{O}(h^{-1})$  would make it possible to run a contraction mapping argument for equations which involve first derivatives – see [HiVa1, HiVa2].

We make the following normally hyperbolic trapping assumptions (see Figure 1):

- (1)  $\Gamma_{\pm}$  are codimension 1 orientable  $C^{\infty}$  submanifolds of  $T^*X$  such that  $\Gamma_{\pm} \cap \{p = 0\} \cap \{q = 0\}$  are compact;
- (2) if  $(x, \xi) \in \{p = 0\} \setminus \Gamma_{\pm}$ , then  $e^{\mp t H_p}(x, \xi) \in \{q > 0\}$  for some  $t \geq 0$ ;
- (3) the Hamiltonian field  $H_p$  is tangent to  $\Gamma_{\pm}$ ;
- (4)  $\Gamma_{\pm}$  intersect transversely,  $K = \Gamma_+ \cap \Gamma_-$  is called the *trapped set* and we assume that  $\text{WF}_h(Q) \cap K \cap \{|p| \leq \delta\} = \emptyset$  for  $\delta > 0$  small enough;
- (5)  $K$  is a symplectic codimension 2 submanifold of  $T^*X$ ;
- (6) if  $v \in T_K \Gamma_{\pm}$ , then  $de^{\mp t H_p} \cdot v$  exponentially approaches  $TK \subset T_K \Gamma_{\pm}$  as  $t \rightarrow +\infty$ .

Assumptions (1)–(6) hold for subextremal Kerr and Kerr–de Sitter black holes with small cosmological constant, see [WuZw, §2], [Va, §6.4], and [Dy2, §3.2].

We furthermore define  $0 < \nu_{\min} \leq \nu_{\max}$  as the maximal and the minimal numbers such that for each  $\varepsilon > 0$  there exists a constant  $C$  such that for each  $v \in T_K \Gamma_{\pm}$

$$C^{-1}e^{-(\nu_{\max}+\varepsilon)t}|\pi_K(v)| \leq |\pi_K(de^{\mp t H_p} \cdot v)| \leq Ce^{-(\nu_{\min}-\varepsilon)t}|\pi_K(v)|, \quad t \geq 0,$$

where  $\pi_K : T_K \Gamma_{\pm} \rightarrow TK$  is any fixed smooth linear projection map. In other words,  $\nu_{\min}$  and  $\nu_{\max}$  are the minimal and maximal expansion rates in directions transversal to the trapped set.

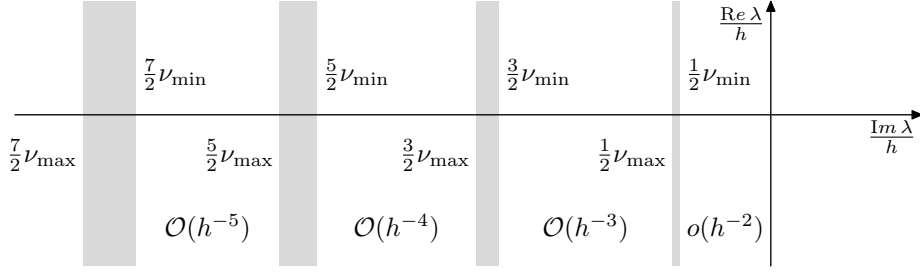


FIGURE 2. The resonance free strips of Theorems 1 and 2 (ignoring  $\varepsilon$ ) are in white, with bounds on the resolvent  $R(\lambda)$  written in the strips and (potential) bands of resonances in between the strips shaded.

**Results.** Our first result is a resonance free strip with a polynomial resolvent bound:

**Theorem 1.** *For each  $\varepsilon > 0$  and  $h$  small enough depending on  $\varepsilon$  (see Figure 2)*

$$\|R(\lambda)\|_{L^2(X) \rightarrow L^2(X)} = o(h^{-2}) \quad \text{if } |\lambda| = \mathcal{O}(h), \quad \text{Im } \lambda > -(\nu_{\min} - \varepsilon)h/2. \quad (1.2)$$

**Remarks.** (i) Theorem 1 also extends to the case when  $P, Q$  are operators acting on sections of some vector bundle  $\mathcal{E}$  over  $X$ , as long as they are self-adjoint with respect to some smooth inner product on the fibers of  $\mathcal{E}$ , and their principal symbols (see for instance [DyZw, §C.1] for a definition) are equal to  $p \cdot \text{Id}_{\mathcal{E}}, q \cdot \text{Id}_{\mathcal{E}}$  for  $p, q \in C^\infty(X)$ . We can also relax the assumption that  $Q = \mathcal{O}(h^\infty)$  microlocally near  $K \cap \{p = 0\}$ , requiring instead that  $q = 0$  and  $\sigma(h^{-1}Q) \geq 0$  near  $K \cap \{p = 0\}$  (as an endomorphism of  $\mathcal{E}$ ). Indeed, the proof of Lemma 3.1 still applies, and the relation (3.9) becomes a lower bound on  $\mu(e^{-tH_p}(U_\delta))$ . We leave the details to the reader.

(ii) We note that polynomial resolvent bounds are known in a variety of other situations – see the review of Wunsch [Wu] for an overview of the subject and [Ch] for other recent results. Also the much earlier work of Gérard–Sjöstrand [GéSj] treated normally hyperbolic trapping in the analytic category with (implicit) exponential resolvent bounds.

We now make the stronger assumption that the trapping is *r-normally hyperbolic* for large  $r$ , namely

$$\nu_{\min} > r\mu_{\max}, \quad (1.3)$$

where  $\mu_{\max}$  is the maximal expansion rate along  $K$ , namely  $\|de^{tH_p}|_{TK}\| = \mathcal{O}(e^{(\mu_{\max} + \varepsilon)t})$  as  $|t| \rightarrow \infty$ , for all  $\varepsilon > 0$ . The condition (1.3) means that the rate of expansion in directions transversal to  $K$  is much larger than the rate of expansion along  $K$ , and it holds for Kerr(–de Sitter) black holes (see the references above).

Under the assumption (1.3) and an appropriate pinching condition, we exhibit deeper resonance free strips with polynomial resolvent bounds:

**Theorem 2.** *Assume that the trapping is  $r$ -normally hyperbolic for all  $r$ . Fix  $m \geq 1$  and assume the pinching condition*

$$(m + 1/2)\nu_{\min} > (m - 1/2)\nu_{\max}.$$

*Then for each small  $\varepsilon > 0$  and  $h$  small enough depending on  $\varepsilon$ , we have (see Figure 2)*

$$\begin{aligned} \|R(\lambda)\|_{L^2(X) \rightarrow L^2(X)} &\leq Ch^{-m-2} \quad \text{if} \\ |\lambda| = \mathcal{O}(h), \quad h^{-1} \operatorname{Im} \lambda &\in [-(m + 1/2)\nu_{\min} + \varepsilon, -(m - 1/2)\nu_{\max} - \varepsilon]. \end{aligned} \quad (1.4)$$

**Remarks.** (i) Since semiclassical arguments only require finitely many derivatives, it is straightforward to see that Theorem 2 is also valid for trapping which is  $r$ -normally hyperbolic for some fixed  $r$ , where  $r$  is chosen large enough depending only on  $m$  and on the dimension of  $X$ . Such trapping is structurally stable under perturbations of the symbol  $p$ , see [HPS] and [Dy1, §5.2].

(ii) Unlike Theorem 1, we only prove Theorem 2 in the scalar case, rather than for operators on general vector bundles, and for  $Q = \mathcal{O}(h^\infty)$  near  $K \cap \{p = 0\}$ .

(iii) The pinching condition is true (for all  $m$ ) for the Schwarzschild(–de Sitter) black hole. In the Kerr case, it was studied for  $m = 1$  in [Dy2, Figure 2(a) and §3.3].

**Ideas of the proofs.** The proof of Theorem 2 is based on constructing pseudodifferential operators  $\Theta_j^+, W_j^+$  such that microlocally near  $K \cap \{p = 0\}$ , we have

$$\Theta_m^+ \dots \Theta_0^+(P - \lambda) = (P - ihW_{m+1}^+ - \lambda)\Theta_m^+ \dots \Theta_0^+ + \mathcal{O}(h^\infty).$$

The principal symbol of  $W_{m+1}^+$  is bounded from below by  $(m + 1)(\nu_{\min} - \varepsilon)$  near  $K \cap \{p = 0\}$ , which has the effect of shifting the spectral parameter  $\lambda$  into the upper half-plane. Since there are no resonances in the upper half-plane, we obtain for  $m$  large enough depending on  $\operatorname{Im} \lambda$ , if  $(P - iQ - \lambda)u = \mathcal{O}(h^\infty)$ , then

$$\Theta_m^+ \dots \Theta_0^+ u = \mathcal{O}(h^\infty) \quad \text{microlocally near } K \cap \{p = 0\}. \quad (1.5)$$

This is a pseudodifferential equation on  $u$ . The principal symbol of each  $\Theta_j^+$  is a defining function  $\varphi_+$  of  $\Gamma_+$ , and the equation (1.5) gives information about the phase space behavior of  $u$  along the Hamiltonian flow lines of  $\varphi_+$ , which are transversal to  $K$  (see Figure 1). Together with the information about the decay of  $u$  as one approaches  $K$ , which depends on  $\operatorname{Im} \lambda$ , we see that  $\lambda$  has to avoid the strips in (1.4).

A good example to keep in mind, considered in §5, is

$$P_0 = \frac{h}{i}(x\partial_x + 1/2).$$

The relevant solutions to the equation  $(P_0 - \lambda)u = 0$  are those whose wavefront set lies on  $\Gamma_+ = \{\xi = 0\}$ , which in particular means that  $u$  has to be smooth. This gives the resonances  $\lambda_m = -(m + \frac{1}{2})ih$ ,  $m = 0, 1, \dots$  with resonant states  $u_m(x) = x^m$  and

the operator  $\partial_x$  maps  $u_m$  to  $mu_{m-1}$ . For the case of  $P_0$ , we can then take  $\Theta_j^+ = \frac{h}{i}\partial_x$  and  $W_j^+ = j$ .

In the case of Theorem 1, we cannot construct the operators  $\Theta_j^+$ , however a rougher version of the operator  $\Theta_0^+$  gives enough information on the concentration of  $u$  along the flow lines of  $H_{\varphi_+}$  to still obtain an estimate. The proof of Theorem 1 is given in §3 and the proof of Theorem 2 is given in §4. Both sections use preliminary dynamical and analytical constructions of §2.

**Bounds on the real line.** For  $\lambda$  on the real line, it is shown in [WuZw] that

$$\|R(\lambda)\|_{L^2 \rightarrow L^2} \leq Ch^{-1} \log(1/h), \quad \text{if } |\lambda| = \mathcal{O}(h), \text{ Im } \lambda = 0. \quad (1.6)$$

The results of Burq–Zworski [BuZw] and Datchev–Vasy [DaVa3] deduce the following improved cutoff resolvent bound:

$$\|R(\lambda)A\|_{L^2 \rightarrow L^2} \leq Ch^{-1} \sqrt{\log(1/h)}, \quad \text{if } |\lambda| = \mathcal{O}(h), \text{ Im } \lambda = 0, \quad (1.7)$$

if  $A$  is an  $h$ -pseudodifferential operator such that  $\text{WF}_h(A) \cap K = \emptyset$ . The bound (1.7) can be improved to  $\mathcal{O}(h^{-1})$  (which is the estimate in the case there is no trapping) if one puts  $A$  on both sides of  $R(\lambda)$ , or if the principal symbol of  $A$  vanishes on  $\Gamma_-$  – see [DaVa2, HiVa3].

In §5, we show that the bound (1.7) is sharp by giving an example of an operator  $P - iQ$  and  $A$  for which the corresponding lower cutoff resolvent bound holds.

## 2. PRELIMINARIES

The proofs in this paper rely on the methods of *semiclassical analysis*. We refer the reader to [Zw] for an introduction to the subject, and to [DyZw, Appendix C.2] for the notation used here. In particular, we use the algebra  $\Psi_h^0(X)$  of semiclassical pseudodifferential operators with symbols in the class  $S^0(X)$ . We moreover require that symbols of the elements of  $\Psi_h^0(X)$  are *classical* in the sense that they have an asymptotic expansion in nonnegative integer powers of  $h$ . Denote by

$$\sigma : \Psi_h^0(X) \rightarrow S^0(X)$$

the principal symbol map and recall the standard identities for  $A, B \in \Psi_h^0(X)$ ,

$$\sigma(AB) = \sigma(A)\sigma(B), \quad \sigma(A^*) = \overline{\sigma(A)}, \quad \sigma(h^{-1}[A, B]) = -i\{\sigma(A), \sigma(B)\}. \quad (2.1)$$

Moreover, each  $A \in \Psi_h^0(X)$  acts  $L^2(X) \rightarrow L^2(X)$  with norm bounded uniformly in  $h$ . In fact, we have the bound [Zw, Theorem 5.1]

$$\limsup_{h \rightarrow 0} \|A\|_{L^2(X) \rightarrow L^2(X)} \leq \|\sigma(A)\|_\infty := \sup_{T^*X} |\sigma(A)|. \quad (2.2)$$

We will mostly use the subalgebra of *compactly microlocalized* pseudodifferential operators,  $\Psi_h^{\text{comp}}(X) \subset \Psi_h^0(X)$ . For  $A \in \Psi_h^{\text{comp}}(X)$ , its *wavefront set*  $\text{WF}_h(A)$  is

a compact subset of  $T^*X$ . We say that  $A = B + \mathcal{O}(h^\infty)$  microlocally in some set  $U \subset T^*X$  if  $\text{WF}_h(A - B) \cap U = \emptyset$ .

We will often consider sequences  $u(h_j) \in L^2(X)$ , where  $h_j \rightarrow 0$  is a sequence of positive numbers; we typically suppress the dependence on  $j$  and simply write  $u = u(h_j)$  and  $h = h_j$ . For such a sequence, we say that  $u = \mathcal{O}(h^\delta)$  *microlocally* in some open set  $U \subset T^*X$ , if  $\|Au\|_{L^2} = \mathcal{O}(h^\delta)$  for each  $A \in \Psi_h^{\text{comp}}(X)$  such that  $\text{WF}_h(A) \subset U$ . The notion of  $u = o(h^\delta)$  in  $U$  is defined similarly.

We may also consider distributions in  $L^2(X; \mathcal{E})$ , where  $\mathcal{E}$  is some smooth vector bundle over  $X$  with a prescribed inner product. One can take pseudodifferential operators acting on sections of  $\mathcal{E}$ , see [DyZw, Appendix C.1], and the principal symbol of such an operator is a section of the endomorphism bundle  $\text{End}(\mathcal{E})$  over  $T^*X$ . However, we will only consider *principally scalar* operators, that is, operators whose principal symbols have the form  $a(x, \xi) \text{Id}_{\mathcal{E}}$ . (Equivalently, these are the operators  $A \in \Psi^k(X; \text{End}(\mathcal{E}))$  such that for all  $B \in \Psi^k(X; \text{End}(\mathcal{E}))$ , we have  $[A, B] = \mathcal{O}(h)$ .) The formulas (2.1) still hold for principally scalar operators. We henceforth suppress  $\mathcal{E}$  in the notation.

**2.1. Semiclassical defect measures.** Our proofs rely on the following definition, designed to capture the concentration of a sequence of  $L^2$  functions in phase space:

**Definition 2.1.** Assume that  $h_j \rightarrow 0$  and  $u = u(h_j) \in L^2(X)$  is bounded uniformly in  $L^2$  as  $j \rightarrow \infty$ . We say that  $u$  converges to a nonnegative Radon measure  $\mu$  on  $T^*X$  if for each  $A = A(h) \in \Psi_h^{\text{comp}}(X)$ , we have

$$\langle A(h_j)u(h_j), u(h_j) \rangle \rightarrow \int_{T^*X} \sigma(A) d\mu \quad \text{as } j \rightarrow \infty. \quad (2.3)$$

See [Zw, Chapter 5] for an introduction to defect measures; in particular,

- for each sequence  $u(h_j)$  uniformly bounded in  $L^2$ , there exists a subsequence  $h_{j_k}$  such that  $u(h_{j_k})$  converges to some  $\mu$  [Zw, Theorem 5.2];
- we have  $\mu(U) = 0$  for some open  $U \subset T^*X$  if and only if  $u = o(1)$  microlocally on  $U$ .

We now show the basic properties of semiclassical measures corresponding to approximate solutions of differential equations:

**Lemma 2.2.** (Ellipticity) Take  $P \in \Psi_h^0(X)$  and denote  $p = \sigma(P)$ . Assume that  $u = u(h_j)$  converges to some measure  $\mu$  and  $Pu = o(1)$  microlocally in some open set  $U \subset T^*X$ . Then  $\mu(U \cap \{p \neq 0\}) = 0$ .

*Proof.* By (2.1), we have for each  $A \in \Psi_h^{\text{comp}}(X)$  with  $\text{WF}_h(A) \subset U$ ,

$$\int_{T^*X} \sigma(A)p d\mu = \lim_{h \rightarrow 0} \langle APu, u \rangle = 0.$$



Since  $\sigma(A)p$  can be any function in  $C_0^\infty(U \cap \{p \neq 0\})$ , we have  $\mu(U \cap \{p \neq 0\}) = 0$ .  $\square$

**Lemma 2.3.** (*Propagation*) Take  $P, W \in \Psi_h^0(X)$ , denote  $p = \sigma(P)$ ,  $w = \sigma(W)$ , and assume that  $P^* = P$ . Assume that  $u = u(h_j)$  converges to some measure  $\mu$  and denote  $f := (P - ihW)u$ . Then for each  $a \in C_0^\infty(T^*X)$  and for each  $Y \in \Psi_h^{\text{comp}}(X)$  such that  $Y = 1 + \mathcal{O}(h^\infty)$  microlocally in a neighborhood of  $\text{supp } a$ ,

$$\left| \int_{T^*X} (H_p - 2 \operatorname{Re} w) a \, d\mu \right| \leq 2 \|a\|_\infty \cdot \limsup_{h \rightarrow 0} (h^{-1} \|Yf\|_{L^2} \cdot \|Yu\|_{L^2}). \quad (2.4)$$

In particular, if  $f = o(h)$  microlocally in a neighborhood of  $\text{supp } a$ , then

$$\int_{T^*X} (H_p - 2 \operatorname{Re} w) a \, d\mu = 0.$$

*Proof.* Take  $A \in \Psi_h^{\text{comp}}(X)$  with  $\sigma(A) = a$  and  $Y = 1 + \mathcal{O}(h^\infty)$  near  $\operatorname{WF}_h(A)$ . Without loss of generality, we may assume that  $W^* = W$ ; indeed, one can put the imaginary part of  $W$  into  $P$ . Then we have

$$\begin{aligned} \frac{\langle Af, u \rangle - \langle Au, f \rangle}{ih} &= \frac{\langle A(P - ihW)u, u \rangle - \langle (P + ihW)Au, u \rangle}{ih} \\ &= \langle ((ih)^{-1}[A, P] - (AW + WA))u, u \rangle. \end{aligned} \quad (2.5)$$

By (2.1),  $\sigma((ih)^{-1}[A, P] - (AW + WA)) = H_p a - 2wa$ ; therefore, the right-hand side of (2.5) converges in absolute value to the left-hand side of (2.4). The left-hand side of (2.5) is equal to

$$\frac{\langle AYf, Yu \rangle - \langle AYu, Yf \rangle}{ih} + \mathcal{O}(h^\infty),$$

and its limit as  $h \rightarrow 0$  is bounded by the right-hand side of (2.4) by (2.2).  $\square$

**2.2. Dynamical preliminaries.** In this section, we review several properties of normally hyperbolic trapping, assuming that  $p, q, \Gamma_\pm, K$  satisfy properties (1)–(6) listed in the introduction. We start with the following restatement of [Dy1, Lemma 5.1], see also [WuZw, Lemma 4.1]. To be able to quantize the functions  $\varphi_\pm, c_\pm$ , we multiply them by a cutoff to obtain compactly supported functions, but only require their properties to hold in a neighborhood  $U$  of  $K \cap \{p = 0\}$ .

**Lemma 2.4.** Fix small  $\varepsilon > 0$ . Then there exists a bounded neighborhood  $U$  of  $K \cap \{p = 0\}$  and functions  $\varphi_\pm \in C_0^\infty(T^*X)$  such that  $\operatorname{WF}_h(Q) \cap U = \emptyset$  and

(1) for  $\delta > 0$  small enough, the set

$$U_\delta := \{|\varphi_+| < \delta, |\varphi_-| < \delta, |p| < \delta\} \cap U \quad (2.6)$$

is compactly contained in  $U$ ;

(2)  $\Gamma_\pm \cap U = \{\varphi_\pm = 0\} \cap U$ ;



(3)  $H_p \varphi_{\pm} = \mp c_{\pm} \varphi_{\pm}$  on  $U$ , where  $c_{\pm} \in C_0^\infty(T^*X)$  satisfy

$$0 < \nu_{\min} - \varepsilon \leq c_{\pm} \leq \nu_{\max} + \varepsilon \quad \text{on } U;$$

(4)  $\{\varphi_+, \varphi_-\} > 0$  on  $U$ .

Note that in particular, for  $\delta > 0$  small enough and all  $t \geq 0$ ,

$$e^{-tH_p}(U_\delta \cap \Gamma_+) \subset \{|\varphi_-| < e^{-(\nu_{\min}-\varepsilon)t}\delta\} \cap U_\delta \cap \Gamma_+. \quad (2.7)$$

Moreover, the map

$$(\rho, s) \mapsto e^{sH_{\varphi_+}}(\rho), \quad \rho \in K, \quad s \in \mathbb{R} \quad (2.8)$$

is a diffeomorphism from some neighborhood of  $(K \cap \{p = 0\}) \times \{s = 0\}$  in  $K \times \mathbb{R}$  onto some neighborhood of  $K \cap \{p = 0\}$  in  $\Gamma_+$ .

For Theorem 2, we also need existence of solutions to the transport equation on  $\Gamma_+$ :

**Lemma 2.5.** *Assume that the flow is  $r$ -normally hyperbolic in the sense of (1.3), for all  $r$ . Fix small  $\delta > 0$  and take  $U_\delta$  defined by (2.6). Then for each  $f \in C^\infty(\Gamma_+ \cap U_\delta)$ , there exists unique  $u \in C^\infty(\Gamma_+ \cap U_\delta)$  such that*

$$(H_p + c_+)u = f. \quad (2.9)$$

*Proof.* We use [Dy1, Lemma 5.2], which gives existence and uniqueness of  $v \in C^\infty(\Gamma_+ \cap U_\delta)$  such that  $H_p v = g$  and  $v|_K = 0$ , for each  $g \in C^\infty(\Gamma_+ \cap U_\delta)$  such that  $g|_K = 0$ .

We claim that there exists a function  $G \in C^\infty(\Gamma_+ \cap U_\delta)$  such that

$$H_p F = c_+ F, \quad F := e^G \{\varphi_+, \varphi_-\}^{-1} \varphi_-.$$

Indeed,  $G$  needs to solve the equation

$$H_p G = c_+ - c_- + \frac{H_p \{\varphi_+, \varphi_-\}}{\{\varphi_+, \varphi_-\}},$$

and we use [Dy1, Lemma 5.2] since the right-hand side of this equation vanishes on  $K$ .

Now (2.9) becomes

$$H_p(Fu) = Ff,$$

and it remains to invoke [Dy1, Lemma 5.2] once again.  $\square$

**2.3. Basic estimates.** We now derive some estimates for the operator  $P - iQ - \lambda$ , where  $P, Q$  satisfy the assumptions of Theorem 1. We start by using the complex absorbing operator  $Q$  to reduce the analysis to a neighborhood of  $K \cap \{p = 0\}$ , where  $K$  is the trapped set:

**Lemma 2.6.** *Assume that  $h_j \rightarrow 0$ ,  $u = u(h_j) \in L^2(X)$  is bounded uniformly in  $h$ , and*

$$\|(P - iQ - \lambda)u\|_{L^2} = \mathcal{O}(h^{\alpha+1})$$

*for some constant  $\alpha > 0$  and  $\lambda = \mathcal{O}(h)$ . Then:*

1.  $u = \mathcal{O}(h^\alpha)$  microlocally on the complement of  $\{p = 0\} \cap \Gamma_+$ .

2. If  $u$  converges to some measure  $\mu$  in the sense of Definition 2.1, then  $\mu$  is supported on  $\{p = 0\} \cap \Gamma_+$ . If moreover  $\|u\|_{L^2} \geq c > 0$  for some constant  $c$ , then for each neighborhood  $U$  of  $\{p = 0\} \cap K$ , we have  $\mu(U) > 0$ .

Same statements are true if we replace  $\mathcal{O}(h^{\alpha+1}), \mathcal{O}(h^\alpha)$  by  $o(h^{\alpha+1}), o(h^\alpha)$  and  $\alpha \geq 0$ .

*Proof.* Take  $A \in \Psi_h^0(X)$  such that  $\text{WF}_h(A) \cap \{p = 0\} \cap \Gamma_+ = \emptyset$ . Then

$$\|Au\|_{L^2} \leq Ch^{-1}\|(P - iQ - \lambda)u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}. \quad (2.10)$$

This follows from the elliptic estimate [DyZw, Proposition 2.4] and propagation of singularities with a complex absorbing operator [DyZw, Proposition 2.5], since for each  $(x, \xi) \in \text{WF}_h(A)$ , there exists  $t \geq 0$  such that  $e^{-tH_p}(x, \xi) \in \{p - iq \neq 0\}$ .

The estimate (2.10) immediately implies part 1 of the lemma, as well as the first statement of part 2. To see the second statement of part 2, it suffices to use the following estimate, valid for each  $B \in \Psi_h^{\text{comp}}(X)$  such that  $\sigma(B) \neq 0$  on  $\{p = 0\} \cap K$ :

$$\|u\|_{L^2} \leq Ch^{-1}\|(P - iQ - \lambda)u\|_{L^2} + \|Bu\|_{L^2}. \quad (2.11)$$

To prove (2.11), we again use the elliptic estimate and propagation of singularities, noting that for each  $(x, \xi) \in T^*X$ , there exists  $t \geq 0$  that  $e^{-tH_p}(x, \xi) \in \{p - iq \neq 0\} \cup \{\sigma(B) \neq 0\}$ , see [Dy1, Lemma 4.1].  $\square$

The next lemma is a generalization of the statement that there are no resonances in the upper half-plane (which, keeping in mind Lemma 2.6, is the special case with  $W = 0$  and  $\text{Im } \lambda \geq ch$ ). See [Dy1, §8.2] for a proof using positive commutator estimates directly instead of going through semiclassical measures.

**Lemma 2.7.** Fix small  $\delta > 0$ ; we use the set  $U_\delta$  defined in (2.6). Assume that  $A, B, B_1 \in \Psi_h^{\text{comp}}(X)$  satisfy (see Figure 3)

- $\text{WF}_h(A) \subset U_{3\delta/2}$  and  $A = 1 + \mathcal{O}(h^\infty)$  microlocally on  $\overline{U_\delta}$ ;
- $\text{WF}_h(B) \subset U_{3\delta} \cap \{|\varphi_+| > \delta/2\}$  and  $B = 1 + \mathcal{O}(h^\infty)$  microlocally on  $\overline{U_{2\delta}} \cap \{|\varphi_+| \geq \delta\}$ ;
- $\text{WF}_h(B_1) \subset U_{3\delta}$  and  $B_1 = 1 + \mathcal{O}(h^\infty)$  microlocally on  $\overline{U_{2\delta}}$ .

Take  $W \in \Psi_h^{\text{comp}}(X)$  and  $\lambda = \mathcal{O}(h)$  such that

$$\text{Re } \sigma(W) + h^{-1} \text{Im } \lambda \geq c > 0 \quad \text{on } U_{3\delta} \quad (2.12)$$

for some constant  $c$ . Then for each  $u \in L^2(X)$ ,

$$\|Au\|_{L^2} \leq Ch^{-1}\|B_1(P - ihW - \lambda)u\|_{L^2} + C\|Bu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}.$$

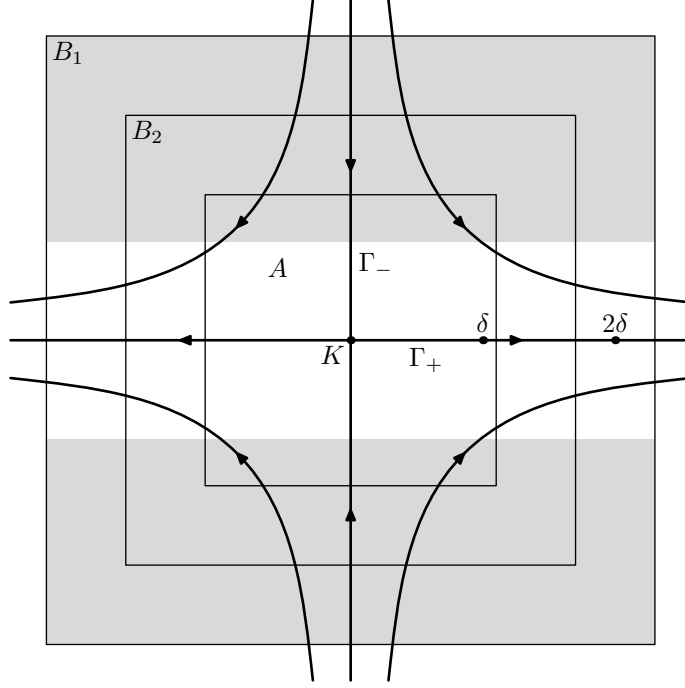


FIGURE 3. The flow  $H_p$  and the wavefront sets of  $A, B, B_1, B_2$ ; here  $\text{WF}_h(B)$  is the shaded portion of  $\text{WF}_h(B_1)$ . The horizontal axis is  $\varphi_-$  and the vertical axis is  $\varphi_+$ .

*Proof.* Take  $B_2 \in \Psi_h^{\text{comp}}(X)$  such that  $\text{WF}_h(B_2) \subset U_{2\delta}$  and  $B_2 = 1 + \mathcal{O}(h^\infty)$  microlocally on  $\overline{U_{3\delta/2}}$ . We first claim that for each fixed  $\varepsilon_0 > 0$  and  $h$  small enough,

$$\|Au\|_{L^2} \leq Ch^{-1}\|B_2(P - ihW - \lambda)u\|_{L^2} + C\|Bu\|_{L^2} + \varepsilon_0\|u\|_{L^2}. \quad (2.13)$$

We argue by contradiction. If (2.13) does not hold, then there exist sequences  $h_j \rightarrow 0$ ,  $\lambda = \lambda(h_j)$ , and  $u = u(h_j) \in L^2$  such that  $\|u\|_{L^2} \leq 1$ ,  $(P - ihW - \lambda)u = o(h)$  microlocally in  $U_{3\delta/2}$ , and  $\|Bu\|_{L^2} = o(1)$ , yet  $\|Au\|_{L^2}$  is separated away from zero.

By passing to a subsequence, we may assume that  $u$  converges to some measure  $\mu$  in the sense of Definition 2.1. Since for each  $(x, \xi) \in U_{3\delta/2} \setminus \Gamma_+$ , there exists  $t \geq 0$  such that  $e^{-tH_p}(x, \xi) \in \{\sigma(B) = 1\}$  and  $e^{-sH_p}(x, \xi) \in U_{3\delta/2}$  for  $s \in [0, t]$ , by propagation of singularities [DyZw, Proposition 2.5] we see that  $u = o(1)$  microlocally in  $U_{3\delta/2} \setminus \Gamma_+$  and thus

$$\mu(U_{3\delta/2} \setminus \Gamma_+) = 0. \quad (2.14)$$

Passing to a subsequence, we may assume that  $h^{-1}\lambda \rightarrow \omega \in \mathbb{C}$ . By Lemma 2.3, for each  $a \in C_0^\infty(U_{3\delta/2})$ ,

$$\int_{T^*X} H_p a \, d\mu = 2 \int_{T^*X} (\text{Re } \sigma(W) + \text{Im } \omega) a \, d\mu.$$

By (2.7), (2.14), and (2.12), it follows that for each nonnegative  $a \in C_0^\infty(U_{3\delta/2})$ ,

$$\int_{\Gamma_+ \cap U_{3\delta/2}} a \circ e^{tH_p} d\mu \geq e^{2ct} \int_{\Gamma_+ \cap U_{3\delta/2}} a d\mu, \quad t \geq 0.$$

However, the left-hand side is bounded uniformly as  $t \rightarrow +\infty$  (by  $\|a\|_\infty \cdot \mu(U_{3\delta/2})$ ), therefore  $\mu(U_{3\delta/2}) = 0$  and thus  $\|Au\|_{L^2} = o(1)$ , giving a contradiction and proving (2.13).

Take  $B'_2 \in \Psi_h^{\text{comp}}(X)$  such that  $B'_2 = 1 + \mathcal{O}(h^\infty)$  microlocally on  $\text{WF}_h(B_2)$ . We apply (2.13) to  $B'_2 u$  to get for each  $\varepsilon_0 > 0$ ,

$$\|Au\|_{L^2} \leq Ch^{-1} \|B_2(P - ihW - \lambda)u\|_{L^2} + C\|Bu\|_{L^2} + \varepsilon_0 \|B'_2 u\|_{L^2} + \mathcal{O}(h^\infty) \|u\|_{L^2}. \quad (2.15)$$

Now, for a correct choice of  $B'_2$ , for each  $(x, \xi) \in \text{WF}_h(B'_2) \cap \{p = 0\}$ , there exists  $t \geq 0$  such that  $e^{-tH_p}(x, \xi) \in \{\sigma(A) \neq 0\} \cup \{\sigma(B) \neq 0\}$  and  $e^{-sH_p}(x, \xi) \in \{\sigma(B_1) \neq 0\}$  for  $s \in [0, t]$ . By propagation of singularities, we then have

$$\|B'_2 u\|_{L^2} \leq Ch^{-1} \|B_1(P - ihW - \lambda)u\|_{L^2} + C\|Bu\|_{L^2} + C\|Au\|_{L^2} + \mathcal{O}(h^\infty) \|u\|_{L^2}. \quad (2.16)$$

It remains to combine (2.15) with (2.16) and take  $\varepsilon_0$  small enough.  $\square$

### 3. FIRST GAP

In this section, we prove Theorem 1. We argue by contradiction. Assume that the estimate (1.2) does not hold; then (replacing  $\varepsilon$  by  $2\varepsilon$ ) there exist sequences  $h_j \rightarrow 0$ ,  $\lambda = \lambda(h_j)$ , and  $u = u(h_j) \in L^2(X)$  such that

$$\|u\|_{L^2} = 1, \quad \|(P - iQ - \lambda)u\|_{L^2} = \mathcal{O}(h^2), \quad |\lambda| = \mathcal{O}(h), \quad \text{Im } \lambda > -(\nu_{\min} - 2\varepsilon)h/2.$$

By Lemma 2.6, we have

$$u = \mathcal{O}(h) \quad \text{microlocally on } T^*X \setminus \Gamma_+. \quad (3.1)$$

Take small  $\delta > 0$  and let  $U_\delta \subset T^*X$  be defined in (2.6). Recall that  $Q = \mathcal{O}(h^\infty)$  microlocally on  $U_{3\delta}$ ; therefore,

$$(P - \lambda)u = \mathcal{O}(h^2) \quad \text{microlocally on } U_{3\delta}. \quad (3.2)$$

Take  $\Theta_+, W_+ \in \Psi_h^{\text{comp}}(X)$  such that

$$\sigma(\Theta_+) = \varphi_+, \quad \sigma(W_+) = c_+, \quad \Theta_+^* = \Theta_+,$$

where  $\varphi_+, c_+$  are constructed in Lemma 2.4.

**Lemma 3.1.** *For  $\text{Im } \lambda > -(\nu_{\min} - 2\varepsilon)h$  (which is weaker than the assumption of Theorem 1),*

$$\Theta_+ u = \mathcal{O}(h) \quad \text{microlocally on } U_\delta. \quad (3.3)$$

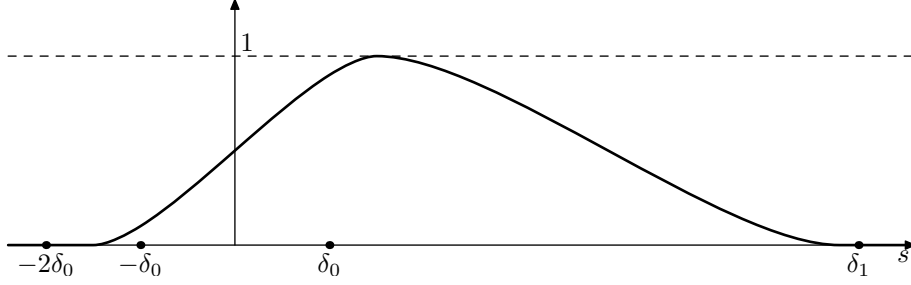


FIGURE 4. The function  $\tilde{a}$  used in the proof of Lemma 3.2. Here  $\delta_1 > 0$  is fixed and  $\delta_0 \rightarrow 0$ .

*Proof.* Since  $H_p\varphi_+ = -c_+\varphi_+$  on  $U_{3\delta}$ , we have

$$[P, \Theta_+] = ihW_+\Theta_+ + \mathcal{O}(h^2)_{\Psi_h^{\text{comp}}} \quad \text{microlocally on } U_{3\delta}. \quad (3.4)$$

Applying  $\Theta_+$  to (3.2) and using (3.4), we obtain

$$(P - ihW_+ - \lambda)\Theta_+u = \mathcal{O}(h^2) \quad \text{microlocally on } U_{3\delta}. \quad (3.5)$$

We have  $\sigma(W_+) = c_+ \geq \nu_{\min} - \varepsilon$  on  $U_{3\delta}$ ; therefore,  $\sigma(W_+) + h^{-1}\text{Im } \lambda \geq \varepsilon > 0$  on  $U_{3\delta}$ . It remains to apply Lemma 2.7 to  $\Theta_+u$  and use (3.1) and (3.5).  $\square$

Now, passing to a subsequence of  $h_j$ , we may assume that (in the sense of Definition 2.1)

$$u \rightarrow \mu, \quad h^{-1}\text{Im } \lambda \rightarrow \omega \in \mathbb{C}.$$

By Lemma 2.6, we have

$$\mu(U_\delta) > 0, \quad \mu(U_\delta \setminus \Gamma_+) = 0. \quad (3.6)$$

We now show that  $\mu$  is Lipschitz in the direction transversal to  $K$ . Recall that  $\varphi_-$ , constructed in Lemma 2.4, is a defining function of  $K$  on  $\Gamma_+ \cap U_\delta$ .

**Lemma 3.2.** *There exists a constant  $C$  such that for each  $\delta_0 > 0$ ,*

$$\mu(U_\delta \cap \{|\varphi_-| < \delta_0\}) \leq C\delta_0. \quad (3.7)$$

*Proof.* We may assume that  $\delta_0$  is arbitrarily small. Applying Lemma 2.3 to (3.3), we see that there exists a constant  $C$  such that for each  $a \in C_0^\infty(\Gamma_+ \cap U_\delta)$ ,

$$\left| \int_{\Gamma_+ \cap U_\delta} H_{\varphi_+} a \, d\mu \right| \leq C\|a\|_\infty. \quad (3.8)$$

To see (3.7), we need to apply this estimate to a function  $a$  depending on  $\delta_0$  which, written in coordinates  $(\rho, s)$  of (2.8), has the form  $a = \tilde{a}(s)\chi(\rho)$  with  $\chi \in C_0^\infty(K \cap \{|p| <$

$\delta\}$ ) equal to 1 on  $K \cap \{p = 0\}$  and

$$\begin{aligned} \text{supp } \tilde{a} &\subset (-2\delta_0, \delta_1), \quad \|\tilde{a}\|_\infty \leq 1, \quad \partial_s \tilde{a} \geq -\frac{2}{\delta_1}, \\ \partial_s \tilde{a} &\geq \frac{1}{3\delta_0} \quad \text{for } |s| \leq \delta_0 \end{aligned}$$

for some fixed small  $\delta_1 > 0$  independent of  $\delta_0$ , see Figure 4. Then

$$\int_{\Gamma_+ \cap U_\delta} H_{\varphi_+} a \, d\mu \geq \frac{1}{3\delta_0} \mu(\{|s| \leq \delta_0\}) - \frac{2}{\delta_1} \mu(\{\delta_0 \leq |s| \leq \delta_1\})$$

The left-hand side is bounded by a  $\delta_0$ -independent constant by (3.8); so is the second term on the right-hand side (since  $\mu$  is a finite measure). Multiplying both sides by  $3\delta_0$  and using that  $\Gamma_+ \cap U_\delta \cap \{|\varphi_-| \leq \delta_0\} \subset \{|s| \leq C\delta_0\}$  for some constant  $C$ , we obtain (3.7).  $\square$

We now have by Lemma 2.3 applied to (3.2), for each  $a \in C_0^\infty(U_\delta)$ ,

$$\int_{\Gamma_+ \cap U_\delta} H_p a \, d\mu = 2 \operatorname{Im} \omega \int_{\Gamma_+ \cap U_\delta} a \, d\mu.$$

For  $t \geq 0$ , since  $e^{-tH_p}(\Gamma_+ \cap U_\delta) \subset \Gamma_+ \cap U_\delta$ , we have

$$\mu(e^{-tH_p}(U_\delta)) = e^{2t \operatorname{Im} \omega} \mu(U_\delta). \quad (3.9)$$

However, by (2.7), (3.6), and (3.7), as  $t \rightarrow +\infty$

$$\mu(e^{-tH_p}(U_\delta)) \leq \mu(U_\delta \cap \{|\varphi_-| < \delta e^{-(\nu_{\min} - \varepsilon)t}\}) \leq C e^{-(\nu_{\min} - \varepsilon)t}.$$

However, since  $\operatorname{Im} \omega \geq -(\nu_{\min} - 2\varepsilon)/2$ , we see that  $e^{2t \operatorname{Im} \omega}$  decays exponentially slower than  $e^{-(\nu_{\min} - \varepsilon)t}$  as  $t \rightarrow +\infty$ . Since  $\mu(U_\delta) > 0$  by (3.6), we arrive to a contradiction, finishing the proof of Theorem 1.

#### 4. FURTHER GAPS FOR $r$ -NORMALLY HYPERBOLIC TRAPPING

In this section, we prove Theorem 2. For small  $\delta > 0$ , let  $U_\delta$  be defined by (2.6).

**4.1. Auxiliary microlocal construction.** The proof relies on the following statement, which in particular makes it possible to improve the remainder in (3.4) to  $\mathcal{O}(h^\infty)$ . For  $Z = 0$ , the operators  $\Theta, W$  below were previously constructed in [Dy1, Proposition 7.1].

**Lemma 4.1.** *Assume that the trapping is  $r$ -normally hyperbolic for all  $r$ , in the sense of (1.3). Then for each  $Z \in \Psi_h^{\text{comp}}(X)$ , there exist  $\Theta, W \in \Psi_h^{\text{comp}}(X)$  such that*

$$\Theta(P - ihZ) = (P - ihW)\Theta + \mathcal{O}(h^\infty) \quad \text{microlocally on } U_{3\delta}; \quad (4.1)$$

$$\sigma(\Theta) = \varphi_+, \quad \sigma(W) = \sigma(Z) + c_+ \quad \text{on } U_{3\delta}. \quad (4.2)$$

*Proof.* We construct operators

$$\Theta_k, W_k \in \Psi_h^{\text{comp}}(X), \quad k = 0, 1, \dots,$$

such that microlocally in  $U_{3\delta}$ , we have for some  $R_k \in \Psi_h^{\text{comp}}(X)$ ,

$$\Theta_k(P - ihZ) = (P - ihW_k)\Theta_k + h^{k+2}R_k + \mathcal{O}(h^\infty) \quad (4.3)$$

and take  $\Theta, W$  to be the asymptotic limits as  $k \rightarrow +\infty$  of these sequences of operators.

We use induction on  $k$ . For  $k = 0$ , take any  $\Theta_0, W_0$  satisfying in  $U_{3\delta}$ ,

$$\sigma(\Theta_0) = \varphi_+, \quad \sigma(W_0) = \sigma(Z) + c_+,$$

then (4.3) follows immediately from (2.1) and the fact that  $H_p\varphi_+ = -c_+\varphi_+$  on  $U_{3\delta}$ .

Assume now that  $k > 0$  and (4.3) holds for  $k - 1$ ; we construct the operators  $\Theta_k$  and  $W_k$  in the form

$$\Theta_k := \Theta_{k-1} + h^k\Omega_k, \quad W_k := W_{k-1} + h^kY_k,$$

where  $\Omega_k, Y_k \in \Psi_h^{\text{comp}}(X)$ . We rewrite (4.3) for  $k$  as the following statement, valid microlocally in  $U_{3\delta}$ :

$$[P, \Omega_k] + ih(\Omega_k Z - W_{k-1}\Omega_k - Y_k\Theta_{k-1}) = hR_{k-1} + \mathcal{O}(h^2)_{\Psi_h^{\text{comp}}}.$$

and this translates to the following equation on the principal symbols<sup>†</sup> of  $\Omega_k, Y_k$  in  $U_{3\delta}$ :

$$(H_p + c_+)\sigma(\Omega_k) + \varphi_+\sigma(Y_k) = i\sigma(R_{k-1}).$$

Since we can choose  $\sigma(Y_k)$  arbitrarily, it suffices to find  $\Omega_k$  whose symbol solves the transport equation

$$(H_p + c_+)\sigma(\Omega_k) = i\sigma(R_{k-1}) \quad \text{on } \Gamma_+ \cap U_{3\delta}.$$

The existence of such a symbol follows from Lemma 2.5, finishing the construction of  $\Theta_k, W_k$ .  $\square$

Iterating Lemma 4.1, we obtain

**Lemma 4.2.** *Assume that the trapping is  $r$ -normally hyperbolic for all  $r$ . Then there exist operators*

$$\Theta_m^+, W_m^+ \in \Psi_h^{\text{comp}}(X), \quad m = 0, 1, \dots$$

such that  $W_+^0 = 0$  and

$$\Theta_m^+(P - ihW_m^+) = (P - ihW_{m+1}^+)\Theta_m^+ + \mathcal{O}(h^\infty) \quad \text{microlocally on } U_{3\delta}. \quad (4.4)$$

Moreover, we have

$$\sigma(\Theta_m^+) = \varphi_+, \quad \sigma(W_m^+) = mc_+ \quad \text{on } U_{3\delta}.$$

---

<sup>†</sup>This is the place where we cannot take  $P$  to act on sections of a vector bundle, as  $R_{k-1}$  and thus  $\Omega_k, Y_k$  need not be principally scalar and a more complicated transport equation would be required.



**4.2. Measures associated to resonant states and proof of Theorem 2.** Assume that  $u = u(h_j) \in L^2(X)$  is bounded uniformly in  $h$  and define  $u_0, u_1, \dots \in L^2(X)$  by the relations

$$u_0 := u, \quad u_{m+1} := \Theta_m^+ u_m. \quad (4.5)$$

**Lemma 4.3.** *Assume that for some  $\alpha \geq 0$  and  $\lambda = \mathcal{O}(h)$ ,*

$$\|(P - iQ - \lambda)u\|_{L^2} = o(h^{\alpha+1}). \quad (4.6)$$

*Then for each  $m \geq 0$ ,*

$$(P - ihW_m^+ - \lambda)u_m = o(h^{\alpha+1}) \quad \text{microlocally on } U_{3\delta}; \quad (4.7)$$

$$u_m = o(h^\alpha) \quad \text{microlocally on } U_{3\delta} \setminus (\Gamma_+ \cap \{p = 0\}). \quad (4.8)$$

*Moreover, if*

$$\operatorname{Im} \lambda > -m(\nu_{\min} - \varepsilon)h + \varepsilon h, \quad (4.9)$$

*then  $u_m = o(h^\alpha)$  microlocally on  $U_\delta$ .*

*Proof.* The condition (4.7) for  $m = 0$  follows directly from (4.6) (since  $Q = \mathcal{O}(h^\infty)$  microlocally on  $U_{3\delta}$ ); for  $m > 0$ , it follows by induction from (4.4). The condition (4.8) for  $m = 0$  follows from Lemma 2.6, and is still true for  $m \geq 0$  since  $u_m$  is obtained by applying an  $h$ -pseudodifferential operator to  $u$ .

The last statement of this lemma follows from (4.7), (4.8) by Lemma 2.7, as on  $U_{3\delta}$ , we have  $\sigma(W_m^+) = mc_+ \geq m(\nu_{\min} - \varepsilon)$  and thus  $\sigma(W_m^+) + h^{-1} \operatorname{Im} \lambda \geq \varepsilon > 0$ .  $\square$

We now obtain information on semiclassical measures of solutions to  $\Theta_m^+ v = o(h)$ , which generalizes the Lipschitz property of Lemma 3.2. It is possible to give a more precise description of the measure along the Hamiltonian flow lines of  $\varphi_+$ ; for  $\Theta_0^+$ , this is done in [Dy1, §8.5, Theorem 4].

**Lemma 4.4.** *Assume that  $v = v(h_j) \in L^2(X)$  converges to some measure  $\mu$  in the sense of Definition 2.1, and  $\mu(U_\delta) > 0$ ,  $\mu(U_\delta \setminus \Gamma_+) = 0$ . Assume also that for some  $m$ ,*

$$\Theta_m^+ v = o(h) \quad \text{microlocally in } U_\delta.$$

*Then for some constant  $C$  and for each  $\delta_0 \in (0, 1)$ , we have*

$$C^{-1}\delta_0 \leq \mu(U_\delta \cap \{|\varphi_-| < \delta_0\}) \leq C\delta_0.$$

*Proof.* Let  $b := \sigma(h^{-1} \operatorname{Im} \Theta_m^+)$ , where we denote  $\operatorname{Im} A := \frac{1}{2i}(A - A^*)$ . By Lemma 2.3, we see that for each  $a \in C_0^\infty(U_\delta)$ ,

$$\int_{U_\delta \cap \Gamma_+} (H_{\varphi_+} + 2b)a \, d\mu = 0.$$

It remains to note that in the coordinates (2.8), we have for each  $a \in C_0^\infty(U_\delta)$

$$\int a d\mu = \int e^{2\psi} a d\tilde{\mu}(\rho) ds$$

where  $\psi \in C^\infty(U_\delta \cap \Gamma_+)$  is some fixed solution to the equation  $H_{\varphi_+} \psi = b$  and  $\tilde{\mu}$  is some measure on  $K$ .  $\square$

Lemma 4.4 makes it possible to show bounds on approximate solutions to  $\Theta_m^+ v = 0$  and  $(P - ihW_m^+ - \lambda)v = 0$  for  $\text{Im } \lambda$  outside of a certain interval:

**Lemma 4.5.** *Let  $A, B_1 \in \Psi_h^{\text{comp}}(X)$  satisfy*

- $\text{WF}_h(A) \subset U_{3\delta/2}$  and  $A = 1 + \mathcal{O}(h^\infty)$  microlocally on  $\overline{U_\delta}$ ;
- $\text{WF}_h(B_1) \subset U_{3\delta}$  and  $B_1 = 1 + \mathcal{O}(h^\infty)$  microlocally on  $\overline{U_{2\delta}}$ .

*Fix  $m$  and assume that  $\lambda = \mathcal{O}(h)$  is such that*

$$h^{-1} \text{Im } \lambda \notin [-(m+1/2)(\nu_{\max} + \varepsilon) - \varepsilon, -(m+1/2)(\nu_{\min} - \varepsilon) + \varepsilon]. \quad (4.10)$$

*Then for each  $v \in L^2(X)$ ,*

$$\|Av\|_{L^2} \leq Ch^{-1} (\|B_1(P - ihW_m^+ - \lambda)v\|_{L^2} + \|B_1\Theta_m^+ v\|_{L^2}) + \mathcal{O}(h^\infty) \|v\|_{L^2}.$$

*Proof.* Arguing similarly to the proof of Lemma 2.7 (removing  $B$  from (2.13) and (2.15) and replacing  $B$  by  $B_1\Theta_m^+$  in (2.16)), we see that it suffices to show that if  $h_j \rightarrow 0$  and  $v = v(h_j)$  satisfies  $\|v\|_{L^2} \leq 1$  and

$$(P - ihW_m^+ - \lambda)v = o(h) \quad \text{microlocally in } U_{3\delta/2}, \quad (4.11)$$

$$\Theta_m^+ v = o(h) \quad \text{microlocally in } U_{3\delta/2}, \quad (4.12)$$

then  $v = o(1)$  microlocally in  $U_{3\delta/2}$ .

Passing to a subsequence, we may assume that  $v$  converges to some measure  $\mu$  in the sense of Definition 2.1 and  $h^{-1}\lambda \rightarrow \omega \in \mathbb{C}$ , where

$$\text{Im } \omega \notin [-(m+1/2)(\nu_{\max} + \varepsilon) - \varepsilon, -(m+1/2)(\nu_{\min} - \varepsilon) + \varepsilon]. \quad (4.13)$$

We have  $\mu(U_{3\delta/2} \setminus \Gamma_+) = 0$  by Lemma 2.2 applied to (4.12). By Lemma 2.3 applied to (4.11), we see that for each  $a \in C_0^\infty(U_{3\delta/2})$ ,

$$\int_{\Gamma_+ \cap U_{3\delta/2}} H_p a d\mu = 2 \int_{\Gamma_+ \cap U_{3\delta/2}} (\text{Im } \omega + \sigma(W_m^+)) a d\mu;$$

for  $t \geq 0$ , since  $e^{-tH_p}(\Gamma_+ \cap U_{3\delta/2}) \subset \Gamma_+ \cap U_{3\delta/2}$ , we have by (4.13) either (when  $\text{Im } \omega$  is too big)

$$\mu(e^{-tH_p}(\Gamma_+ \cap U_{3\delta/2})) \geq e^{-(\nu_{\min} - 3\varepsilon)t} \mu(U_{3\delta/2}) \quad (4.14)$$

or (when  $\text{Im } \omega$  is too small)

$$\mu(e^{-tH_p}(\Gamma_+ \cap U_{3\delta/2})) \leq e^{-(\nu_{\max} + 3\varepsilon)t} \mu(U_{3\delta/2}). \quad (4.15)$$

Here we used that  $\sigma(W_m^+) = mc_+ \in [m(\nu_{\min} - \varepsilon), m(\nu_{\max} + \varepsilon)]$  on  $U_{3\delta/2}$ . Using Lemma 4.4 with (4.12) and observing that for  $t \geq 0$ , by Lemma 2.4

$$\begin{aligned} \{|\varphi_-| < \tfrac{3\delta}{2}e^{-(\nu_{\max}+\varepsilon)t}\} \cap \Gamma_+ \cap U_{3\delta/2} &\subset e^{-tH_p}(\Gamma_+ \cap U_{3\delta/2}), \\ e^{-tH_p}(\Gamma_+ \cap U_{3\delta/2}) &\subset \{|\varphi_-| < \tfrac{3\delta}{2}e^{-(\nu_{\min}-\varepsilon)t}\} \cap \Gamma_+ \cap U_{3\delta/2} \end{aligned}$$

both (4.14) and (4.15) imply, by taking the limit  $t \rightarrow +\infty$ , that  $\mu(U_{3\delta/2}) = 0$  and thus  $v = o(1)$  microlocally on  $U_{3\delta/2}$ , finishing the proof.  $\square$

We can now prove Theorem 2. We argue by contradiction. If (1.4) does not hold, then (changing  $\varepsilon$ ) there exist sequences  $h_j \rightarrow 0$  and  $u = u(h_j) \in L^2(X)$  such that

$$\|u\|_{L^2} = 1, \quad \|(P - iQ - \lambda)u\|_{L^2} = o(h^{m+2}),$$

where  $\lambda = \lambda(h_j) = \mathcal{O}(h_j)$  satisfies

$$h^{-1} \operatorname{Im} \lambda \in (- (m + 1/2)(\nu_{\min} - \varepsilon) + \varepsilon, - (m - 1/2)(\nu_{\max} + \varepsilon) - \varepsilon). \quad (4.16)$$

Define  $u_0, \dots, u_{m+1}$  by (4.5); then by Lemma 4.3,  $u_{m+1} = o(h^{m+1})$  microlocally in  $U_\delta$ . Here we used the following corollary of (4.16):

$$h^{-1} \operatorname{Im} \lambda > - (m + 1)(\nu_{\min} - \varepsilon) + \varepsilon.$$

Using Lemma 4.5 for  $v = u_m, u_{m-1}, \dots, u_0$  (and a decreasing sequence of  $\delta$ 's), we get  $u_j = o(h^j)$  microlocally in  $U_{3^{j-m-1}\delta}$  for  $j = 0, 1, \dots, m+1$ . Here we used the following corollary of (4.16):

$$h^{-1} \operatorname{Im} \lambda \notin [-(j + 1/2)(\nu_{\max} + \varepsilon) - \varepsilon, -(j + 1/2)(\nu_{\min} - \varepsilon) + \varepsilon], \quad 0 \leq j \leq m.$$

In particular, we see that  $u = u_0 = o(1)$  microlocally in the neighborhood  $U_{3^{-m-1}\delta}$  of  $K \cap \{p = 0\}$ , giving a contradiction with Lemma 2.6 and finishing the proof of Theorem 2.

## 5. A ONE-DIMENSIONAL EXAMPLE

In this section, we show a lower bound on the cutoff resolvent for  $\lambda = 0$  and a specific operator with normally hyperbolic trapping. We start with the model operator (which we later put into the framework discussed in the introduction)

$$P_0 = xhD_x + \frac{h}{2i} : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \quad D := \frac{1}{i}\partial.$$

Note that  $P_0^* = P_0$ . Take a function  $\chi \in \mathcal{S}(\mathbb{R})$  such that  $\hat{\chi} \in C_0^\infty((1/2, 1))$  and define

$$\psi(x) = \operatorname{sgn} x \int_0^x |x|^{-1/2} |y|^{-1/2} \chi(y) dy.$$

Then (as can be seen by considering the Taylor expansion of  $\chi$  at 0) we have

$$\psi \in C^\infty(\mathbb{R}), \quad P_0\psi = -ih\chi.$$

Moreover, we have

$$\psi(x) = \psi_{\pm}|x|^{-1/2} + \mathcal{O}(|x|^{-\infty}) \quad \text{as } x \rightarrow \pm\infty, \quad (5.1)$$

where the constants  $\psi_{\pm}$  are given by

$$\psi_{\pm} = \pm \int_0^{\pm\infty} |y|^{-1/2} \chi(y) dy.$$

We choose  $\chi$  so that  $\psi_{\pm} \neq 0$ ; this is possible since

$$\psi_{\pm} = \frac{e^{\pm i\pi/4}}{2\sqrt{\pi}} \int_0^{\infty} \xi^{-1/2} \hat{\chi}(\xi) d\xi.$$

Define

$$u_0(x; h) := h^{-1/2} \psi(x/h), \quad f_0(x; h) := -ih^{1/2} \chi(x/h),$$

then  $P_0 u_0 = f_0$ . We have

$$\text{WF}_h(u_0) \subset \{\xi = 0\} \cup \{x = 0, \xi \in [0, 1]\}. \quad (5.2)$$

Indeed, if  $a \in C_0^{\infty}((0, \infty))$ , then by (5.1),

$$a(x)u_0(x) = \psi_+ a(x)x^{-1/2} + \mathcal{O}(h^{\infty});$$

since  $\psi_+ a(x)x^{-1/2}$  is smooth and independent of  $h$ , we see by [Zw, §8.4.2] that  $\text{WF}_h(au_0) \subset \{\xi = 0\}$ . Same is true when  $a \in C_0^{\infty}((-\infty, 0))$ . Moreover, since  $(x\partial_x + \frac{1}{2})\psi = \chi$ , we have  $(\xi\partial_{\xi} + \frac{1}{2})\hat{\psi} = -\hat{\chi}$ ; since  $\partial_x \psi \in L^2$ , we have  $\xi\hat{\psi}(\xi) \in L^2$  and the fact that  $\text{supp } \hat{\chi} \subset (1/2, 1)$  implies  $\text{supp } \hat{\psi} \subset [0, 1)$ . This in turn implies that  $\text{WF}_h(u_0) \subset \{\xi \in [0, 1]\}$ , finishing the proof of (5.2). Since  $\text{supp } \hat{\chi} \subset (1/2, 1)$  and  $\chi$  is Schwartz, we also have

$$\text{WF}_h(f_0) \subset \{x = 0, \xi \in (1/2, 1)\}. \quad (5.3)$$

We furthermore calculate using (5.1)

$$\|u_0\|_{L^2(-1,1)} = \sqrt{|\psi_+|^2 + |\psi_-|^2} \sqrt{\log(1/h)} + \mathcal{O}(1), \quad \|f_0\|_{L^2(\mathbb{R})} = \mathcal{O}(h). \quad (5.4)$$

We now put  $u_0, f_0, P_0$  into the framework discussed in the introduction. Put

$$X = \mathbb{S}^1 = \mathbb{R}/(6\mathbb{Z});$$

we view  $X$  as the interval  $[-3, 3]$  with the endpoints glued together. We consider an operator

$$P \in \Psi_h^{\text{comp}}(X), \quad P = P^*; \quad P = P_0 \quad \text{microlocally on } \{|x| \leq 2, |\xi| \leq 2\}.$$

Note that by (5.2), we have  $Pu_0 = f_0 + \mathcal{O}(h^{\infty})_{C^{\infty}}$  on  $(-2, 2)$ .

We next consider  $q_0 \in C^{\infty}(X)$  such that  $q_0 = 0$  on  $[-1, 1]$ ,  $q_0 \geq 0$  everywhere, and  $q_0 = 1$  outside of  $[-3/2, 3/2]$ . Take  $\chi_1 \in C_0^{\infty}(-2, 2)$  such that  $\chi_1 = 1$  on  $[-3/2, 3/2]$  and define

$$u_1 = \chi_1 u_0, \quad f_1 = (P - iq_0)u_1 - \chi_1 f_0 = [P, \chi_1]u_0 - iq_0 \chi_1 u_0 + \mathcal{O}(h^{\infty})_{C^{\infty}}.$$

In particular,  $f_1 = \mathcal{O}(h^\infty)$  on  $[-1, 1]$  and  $\text{WF}_h(f_1) \subset \{\xi = 0\}$ . Using ODE theory for the operator  $P_0 - iq_0 = \frac{h}{i}(x\partial_x + \frac{1}{2} + h^{-1}q_0)$ , we construct  $u_2 \in C^\infty(X)$  such that  $u_2 = 0$  on  $[-1, 1]$ ,  $u_2 = \mathcal{O}(e^{-\varepsilon/h})$  outside of  $[-7/4, 7/4]$ ,  $\text{WF}_h(u_2) \subset \{\xi = 0\}$ , and

$$(P - iq_0)u_2 = -f_1 + \mathcal{O}(h^\infty)_{C^\infty}.$$

Finally, take a self-adjoint operator  $Q_1 \in \Psi_h^0$  such that  $\text{WF}_h(Q_1) \subset \{|\xi| \geq 3/2\}$ ,  $\sigma(Q_1) \geq 0$  everywhere, and  $\sigma(Q_1) > 0$  for  $|\xi| \geq 2$ , denote by  $Q_0$  the multiplication operator by  $q_0$ , and put

$$Q := Q_0 + Q_1, \quad \sigma(Q) = q_0 + \sigma(Q_1).$$

Then the operator  $P - iQ : L^2(X) \rightarrow L^2(X)$  satisfies the assumptions (1)–(6) of the introduction, with  $\nu_{\min} = \nu_{\max} = 1$  and

$$\Gamma_+ = \{\xi = 0, |x| < 2\}, \quad \Gamma_- = \{x = 0, |\xi| < 2\}, \quad K = \{x = \xi = 0\}.$$

We have

$$(P - iQ)u = \chi_1 f_0 + \mathcal{O}(h^\infty), \quad u := u_1 + u_2.$$

On the other hand, (5.4) implies

$$\|u\|_{L^2} \geq C^{-1} \sqrt{\log(1/h)}, \quad \|\chi_1 f_0\|_{L^2} = \mathcal{O}(h).$$

This means that for each  $A \in \Psi_h^{\text{comp}}(X)$  such that

$$A = 1 + \mathcal{O}(h^\infty) \quad \text{microlocally on } V := \{x = 0, \xi \in (1/2, 1)\},$$

we have

$$\|R(0)A\|_{L^2 \rightarrow L^2} \geq C^{-1} h^{-1} \sqrt{\log(1/h)}.$$

Note that  $V \subset \Gamma_-$  and  $V \cap \Gamma_+ = \emptyset$ , therefore we can choose  $A$  to be microlocalized away from the trapped set  $K$ .

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